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Exponentiation of the Drell-Yan cross section near partonic threshold in the DIS and $\overline{\rm MS}$ schemes

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Abstract: It has been observed that in the DIS scheme the refactorization of the Drell-Yan cross section leading to exponentiation of threshold logarithms can also be used to organize a class of constant terms, most of which arise from the ratio of the timelike Sudakov form factor to its spacelike counterpart. We extend this exponentiation to include *all* constant terms, and demonstrate how a similar organization may be achieved in the $\overline{\rm MS}$ scheme. We study the relevance of these exponentiations in a two-loop analysis.

Keywords: Renormalization Group, QCD, Deep Inelastic Scattering, Hadronic Colliders.

Contents

1. Introduction

The predictive power of QCD for high-energy hadronic scattering observables rests upon our ability to compute the corresponding partonic cross sections in perturbation theory. Factorization theorems [1] assert that for many such cross sections mass divergences may be subtracted, or mass-factorized, in a process-independent way, with any additional finite subtraction constants defining the mass-factorization scheme.

In many cases, the predictive power of such perturbative series is imperiled by the systematic occurrence of finite but large terms at higher orders. Resummation attempts to restore predictive power by organizing classes of large terms to all orders, leaving a perturbative series for the remainder with much better convergence properties [2]. For cross sections, the large terms almost always take the form of logarithms of ratios of kinematical scales. In particular, threshold resummations [3, 4] organize logarithmic enhancements singular at partonic threshold, resulting from imperfect cancellations between real and purely virtual contributions to the cross section. As partonic threshold is approached, these enhancements are parametrically guaranteed to increasingly dominate the perturbative contributions to the cross section. Often, however, also constant terms, which do not depend on scale ratios vanishing at threshold and which arise predominantly from purely virtual diagrams, are numerically important in the cross section. Some of these large constants originate from the same infrared singularities that give rise to the large logarithms, and consequently are resummable.

Threshold resummations can be derived for partonic cross sections using the procedure of refactorization: the Mellin transform of the cross section is expressed near threshold as a product of well-defined functions, each organizing a class of infrared and collinear enhance-

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ments. The refactorizations are valid up to corrections which are nonsingular at threshold, and thus suppressed by powers of the Mellin variable N at large-N. Terms independent of N can then be treated, in principle, by the same methods used to resum terms enhanced by logarithms of N. For the Drell-Yan cross section and the deep-inelastic structure function F_2 such a refactorization was achieved in ref. [3]. The resulting resummation of constant terms established to all orders the earlier observation [5] that, in the DIS-scheme Drell-Yan cross section, the largest constants are related to the ratio of the timelike Sudakov form factor to its spacelike continuation. By solving an appropriate evolution equation [6], an exponential representation for this form factor, and thus also for the ratio, was derived [7], generalizing the observation of [5] to a full nonabelian exponentiation. Comparison with exact two-loop results [8] showed in that case that N-independent contributions at two loops are indeed dominated by the exponentiation of the one-loop result, combined with running coupling effects [9, 10].

In this paper we refine this analysis for the DIS scheme, and we extend it to the $\overline{\text{MS}}$ scheme. The challenge in the latter case lies in the fact that the finite subtraction constants of this scheme are not related to a physical scattering process involving the electromagnetic quark coupling at lowest order. Therefore, ratios of form factors do not naturally occur in the $\overline{\text{MS}}$ scheme. The practical prevalence and relative simplicity of the $\overline{\text{MS}}$ scheme would, however, make such an organization desirable. We will show that the refactorization formalism of ref. [3] leads to the exponentiation of *all* N-independent contributions to the inclusive Drell-Yan cross section, both in the $\overline{\text{MS}}$ scheme and in the DIS scheme. As a corollary, one may note that all constant terms in the $\overline{\text{MS}}$ -scheme non-singlet deep-inelastic structure function F_2 have been organized into an exponential form as well. Furthermore, it is possible to organize the factorization procedure so that real and virtual contributions are individually made finite; one can then disentangle various sources of constants, such as π^2 terms arising from unitarity cuts and similar terms arising from expansions of phase-space related Γ functions.

One might object that there is no kinematic limit in which N-independent terms dominate parametrically, so that an organization of such terms cannot be of much practical use. In general, our view is that whenever all-order information is available one should make use of it, at least to gauge the potential impact of generic higher order corrections on the cross section at hand. One should bear in mind that the pattern of exponentiation, even for N-independent terms, is highly nontrivial, and includes all-order information arising from renormalization group evolution and the requirements of factorization; for example, a considerable fraction of nonabelian effects arising in the Sudakov form factor at two loops can be shown to follow from running coupling effects implemented on the (essentially abelian) one-loop result. In the present case one could be bolder and argue that, since all constants have been shown to exponentiate, using the exponentiated expression should provide a better approximation to the exact answer. This is in fact the case for generic values of higher order perturbative coefficents, even for asymptotic series such as those arising in QCD. It cannot, however, be proven for any particular cross section, although it works in practice for the cases that have been tested. At the very least, differences between results for the physical cross section with or without the exponentiation of constant terms can provide nontrivial estimates of errors due to (uncalculated) higher order corrections. Notice in passing that constant terms are not affected by the Landau pole and thus factor out of the inverse Mellin transform needed to construct the physical cross section.

This paper is organized as follows. In the next section we review the resummation of the quark Sudakov form factor and its embedding in the DIS-scheme Drell-Yan cross section. We extend the exponentiation in this scheme to include all constant terms. In section 3 we derive our result for the $\overline{\rm MS}$ cross section. In section 4 we compare the results of the exponentiations to finite order calculations, while section 5 contains our conclusions. In an appendix we present renormalization group studies of certain auxiliary functions.

2. Exponentiation in the DIS scheme

Consider the N-th moment of the partonic Drell-Yan cross section, taken with respect to $z = Q^2/s$, with Q the measured invariant mass and s the partonic invariant mass squared. Mass factorization of this quantity, in the DIS scheme, is performed by simply dividing its dimensionally regularized ($d = 4 - 2\epsilon$), unsubtracted version by the square of the N-th moment (taken with respect to the partonic Bjorken-x variable) of the non-singlet partonic deep-inelastic structure function F_2,

$$\widehat{\omega}_{\rm DIS}(N) = \frac{\omega(N,\epsilon)}{[F_2(N,\epsilon)]^2}. \tag{2.1}$$

While numerator and denominator are each infrared and collinear divergent, their ratio is finite to all perturbative orders [1], so the ϵ dependence of the left hand side can be neglected. Dependence on the hard scale Q and on the strong coupling $\alpha_s(\mu^2)$ will generally be understood.

Let us begin by identifying how the Sudakov form factor arises in $\omega(N,\epsilon)$, before mass factorization, following the reasoning of ref. [3]. One observes that near threshold (i.e. at large-N) $\omega(N,\epsilon)$ can be refactorized according to

$$\omega(N,\epsilon) = |H_{\rm DY}|^2\, \psi(N,\epsilon)^2 U(N) + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.2}$$

The $\psi(N,\epsilon)$ and $U(N)$ functions contain the N dependence associated with initial state radiation at fixed energy and coherent soft radiation, respectively. They are well-defined as operator matrix elements, as given in ref. [3], and are calculable in perturbation theory. Both functions are gauge-dependent, but their product in eq. (2.2) is not; implicitly, we will be working in axial gauge. Using gauge invariance and renormalization group (RG) arguments, one can show that both the parton distribution $\psi(N,\epsilon)$ and the eikonal function $U(N)$ obey evolution equations which can be solved near threshold in an exponential form, up to corrections suppressed by powers of N. The function $|H_{\rm DY}|^2$, collecting all hard-gluon corrections, has no N dependence and may be determined by matching to exact calculations order by order. Divergences are only present in the parton distribution function $\psi(N,\epsilon)$.

To identify the Sudakov form factor, it is useful [3] to separate virtual (V) and real (R) contributions to the resummed ψ and U functions, according to

$$\begin{aligned} \psi(N,\epsilon) &= \mathcal{R}(\epsilon)\,\psi_R(N,\epsilon) \\ U(N) &= U_V(\epsilon)\,U_R(N,\epsilon)\,, \end{aligned} \tag{2.3}$$

where $\mathcal{R}(\epsilon)$ is the real part of the residue of the quark two-point function in an axial gauge. Note that this further factorization makes sense only because the functions ψ and U are known to exponentiate to the present accuracy. Using the analysis of section 8 of ref. [3] we can now write

$$\begin{aligned} \omega(N,\epsilon) &= |H_{\rm DY}\,\mathcal{R}(\epsilon)\sqrt{U_V(\epsilon)}|^2\,\psi_R(N,\epsilon)^2\,U_R(N) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= |\Gamma(Q^2,\epsilon)|^2\,\psi_R(N,\epsilon)^2\,U_R(N,\epsilon) + \mathcal{O}\left(\frac{1}{N}\right), \end{aligned} \tag{2.4}$$

so that all virtual contributions are expressed in terms of the electromagnetic quark form factor $\Gamma(Q^2,\epsilon)$. In fact, the residue of the quark propagator coincides with the virtual jet function summarizing virtual collinear contributions to the form factor, while the square root of the virtual eikonal function appearing in the cross section is responsible for the soft enhancements of $\Gamma(Q^2,\epsilon)$. We have thus indentified the dimensionally regularized time-like Sudakov form factor in the refactorized, unsubtracted Drell-Yan cross section. Near threshold, the only remaining contributions to the cross section come from real radiation, and are summarized by the real parts of the ψ and U functions. At this point one can already observe that $\omega_{\rm DY}(N,\epsilon)$ exponentiates up to corrections suppressed by powers of N: the exponentiation of the form factor in dimensional regularization was proven in ref. [7], while the exponentiation of ψ_R and U_R to this accuracy was proven in ref. [3]. Specifically, the real part of the fixed-energy parton density $\psi_R(N,\epsilon)$ can be written as

$$\psi_R(N,\epsilon) = \exp\left\{\int_0^1 dz\,\frac{z^{N-1}}{1-z}\int_z^1\frac{dy}{1-y}\;\kappa_\psi\left(\overline{\alpha}\left((1-y)^2Q^2\right),\epsilon\right)\right\}. \tag{2.5}$$

Similarly

$$U_R(N,\epsilon) = \exp\left\{-\int_0^1 dz\,\frac{z^{N-1}}{1-z}\,g_U\left(\overline{\alpha}\left((1-z)^2Q^2\right),\epsilon\right)\right\}. \tag{2.6}$$

Note that in writing eq. (2.5) and eq. (2.6) one makes use of the fact that both functions are renormalization-group invariant, since real emission diagrams do not have in this case ultraviolet divergences. For the function ψ_R this is a consequence of the fact that it fixes the energy of the final state, so that transverse momentum is also limited; for the function U_R it is a consequence of the structure of nonabelian exponentiation, as discussed in ref. [3]. The functions κ_ψ and g_U are both finite at one loop in the limit $\epsilon \to 0$: all IR and collinear singularities are generated by the integrations over the scale of the d-dimensional running coupling, which at one loop is given by

$$\overline{\alpha}\left(\frac{\xi^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \alpha_s(\mu^2)\left[\left(\frac{\xi^2}{\mu^2}\right)^{2\epsilon} - \frac{1}{\epsilon}\left\{1-\left(\frac{\xi^2}{\mu^2}\right)^{2\epsilon}\right\}\frac{b_0}{4\pi}\alpha_s(\mu^2)\right]^{-1}, \tag{2.7}$$

and will often be abbreviated by $\overline{\alpha}(\xi^2)$, as in eqs. (2.5) and (2.6).

A similar refactorization can be performed on the deep inelastic structure function F_2 [3]. One finds

$$F_2(N,\epsilon) = |H_{\rm DIS}|^2\, \chi(N,\epsilon)\, V(N)\, J(N) + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.8}$$

Here the parton distribution $\chi(N,\epsilon)$ has the same soft and collinear singularities as the one adopted for the Drell-Yan process, but, according to the general strategy of ref. [3], it fixes a different component of the incoming parton momentum (the plus-component) and is computed in a different axial gauge. It exponentiates in a form similar to eq. (2.5), but with a different function κ_χ appearing in the exponent. Note that κ_ψ and κ_χ must (and do) differ at one loop only by terms of order ϵ^2, so that the ratio ψ_R/χ_R may remain finite. The soft function $V(N)$ summarizes the effects of coherent soft radiation in the DIS process, and exponentiates in a form similar to eq. (2.6). Finally, $J(N)$ contains the effects of final state collinear radiation emitted by the struck parton, and separately exponentiates in a somewhat more elaborate form which will not be needed here. Separating real and virtual contributions as above we find

$$\begin{aligned} F_2(N,\epsilon) &= |H_{\rm DIS}|^2\, |\mathcal{R}(\epsilon) V_V(\epsilon)|\, \chi_R(N,\epsilon)\, V_R(N,\epsilon)\, J(N) + \mathcal{O}\left(\frac{1}{N}\right) \\ &= |H_{\rm DIS}| |\Gamma(-Q^2,\epsilon)|\, \chi_R(N,\epsilon)\, \sqrt{V_V(\epsilon)}\, V_R(N,\epsilon)\, J(N) + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned} \tag{2.9}$$

To this extent virtual contributions to F_2 have been organized already in [3]. We now see that *all* virtual contributions can be organized in terms of $\Gamma(-Q^2,\epsilon)$, by observing that the purely virtual part of the light-cone distribution χ is identical to the purely virtual part of the outgoing jet J. Both consist essentially of the full two-point function for a lightlike fermion. Note that both virtual jets in eq. (2.9) are computed with the same gauge choice. Gathering all virtual parts, one finds then

$$F_2(N,\epsilon) = |\Gamma(-Q^2,\epsilon)|^2\, \chi_R(N,\epsilon)\, V_R(N,\epsilon)\, J_R(N,\epsilon) + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.10}$$

Again, exploiting the results of refs. [3, 7] this form of the refactorization is sufficient to prove the exponentiation of the full cross section up to corrections suppressed by powers of N. In fact F_2 now involves, to this accuracy, only the form factor, and a product of real functions which have been shown to exponentiate by using their respective evolution equations.

This result can be further verified in the following way. A comparison of eqs. (2.9) and (2.10) implies that $H_{\rm DIS}$ itself acquires an exponential form. In fact, an analysis along the lines of [6] reveals that it may be expressed as

$$H_{\rm DIS}(Q^2) = Z_H(\alpha_s,\epsilon)\, \frac{\Gamma(-Q^2,\epsilon)}{S^{(0)}(\epsilon)\; G_2^{(0)}(Q^2,\epsilon)}\,, \tag{2.11}$$

where $S^{(0)} = S\, Z_S$ and $G_2^{(0)} = G_2 Z_q$ are the unrenormalized, dimensionally regularized virtual soft function and virtual quark jet function appearing in the factorization of the form factor, and where

$$Z_H = Z_S \, Z_q^2 \tag{2.12}$$

is the UV counterterm function for $H_{\rm DIS}$. The Sudakov form factor Γ does not require a separate Z factor to cancel QCD UV divergences, by virtue of electromagnetic current conservation. Now, each factor in eq. (2.11) has an exponential form. For the virtual soft function this was shown by Gatheral, Frenkel and Taylor in [11, 12]. The unrenormalized virtual jet function $G_2^{(0)}(Q^2, \epsilon)$ obeys an evolution equation of the same form as the one used for the full form factor [6], which can be explicitly solved in dimensional regularization by the same methods, using as initial condition the fact that all radiative corrections vanish at $Q^2 = 0$. $G_2^{(0)}(Q^2, \epsilon)$ must then exponentiate by itself. Finally, any Z factor arising in multiplicative renormalization may be represented in a minimal scheme in terms of the associated anomalous dimension $\gamma = (1/2) d(\ln Z)/d\ln\mu$ as

$$Z_i = \exp\left\{\int_0^{Q^2} \frac{d\xi^2}{\xi^2} \gamma_i\left(\overline{\alpha}\left(\xi^2\right)\right)\right\}, \tag{2.13}$$

where again UV poles are generated by integration over the scale of the d-dimensional coupling.

Turning to the evaluation of eq. (2.1), we observe that it requires the ratio of eq. (2.4) and the square of either eq. (2.9) or (2.10). In practice, the expression (2.9) is more convenient than eq. (2.10), because the resulting form (2.15) for $\widehat{\omega}_{\rm DIS}(N)$ is a product of finite functions. Had one used instead the result in eq. (2.10), the resulting expression for $\widehat{\omega}_{\rm DIS}(N)$ would have involved cancelling divergences between the real and virtual parts. Using then eq. (2.9), and the additional information that $U_V(\epsilon) = V_V(\epsilon)$, both being given by pure counterterms to the same eikonal vertex, we can write

$$\widehat{\omega}_{\rm DIS}(N) = \frac{1}{|H_{DIS}|^2}\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \left(\frac{\psi_R(N,\epsilon)}{\chi_R(N,\epsilon)}\right)^2 \frac{U(N)}{V^2(N)} \frac{1}{J^2(N)}\,. \tag{2.14}$$

The exponentiation and RG running of the various factors of eq. (2.14) are described in detail in [3], with the exception of the ratio ψ_R/χ_R, which there was exponentiated according to eq. (2.5), but evaluated only at leading order. Running coupling effects on this ratio are briefly discussed in an appendix: they generate a contribution at NNL log level at two loops, as well as further N-independent terms. Furthermore, we are now in a position to exponentiate also the one-loop contribution to the matching function, setting $H_{DIS} = \exp\left(-\alpha_s C_F/\pi\right)$. Gathering all factors, and formulating the answer according to standard notation [4], our result for the hard partonic Drell-Yan cross section in the DIS scheme takes the form

$$\begin{aligned}\widehat{\omega}_{\rm DIS}(N) \; = \; & \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \exp\Big[F_{DIS}(\alpha_s)\Big] \\ & \times \exp\left[\int_0^1 dz\, \frac{z^{N-1}-1}{1-z}\left\{2\int_{(1-z)Q^2}^{(1-z)^2 Q^2} \frac{d\xi^2}{\xi^2} A\left(\alpha_s(\xi^2)\right) - 2B\left(\alpha_s\left((1-z)Q^2\right)\right) + \right.\right. \\ & \left.\left. + D\left(\alpha_s\left((1-z)^2 Q^2\right)\right)\right\}\right] + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned} \tag{2.15}$$

Eq. (2.15) resums all terms in the perturbative expansion which contain enhancements of the form $\alpha_s^n \log^k N$, with $k \leq 2n$, provided the perturbative expansions of the functions A, B and D are known to the desired order in the strong coupling. The perturbative coefficients of the functions A, B, D are in fact all known up to two loops [13, 14, 15].[1] For example, expanding the functions involved as

$$f(\alpha_s) = \sum_{k=1}^{\infty} f^{(k)} \left(\frac{\alpha_s}{\pi}\right)^k , \tag{2.16}$$

one needs

$$\begin{aligned} A^{(1)} &= C_F \, , \qquad A^{(2)} = \frac{1}{2}\left[C_A C_F \left(\frac{67}{18} - \zeta(2)\right) - n_f C_F \left(\frac{5}{9}\right)\right] , \\ B^{(1)} &= -\frac{3}{4} C_F \, , \qquad D^{(1)} = 0 \, . \end{aligned} \tag{2.17}$$

for resummation to next-to-leading logarithmic accuracy.

Eq. (2.15) also exponentiates N-independent terms, which have three sources: they come from unitarity cuts, as in the analytic continuation of the form factor; or from phase space integrations, since for example the parton distributions ψ and χ differ slightly in their phase space measure; finally they arise from the Mellin transformation in the exponent, which generates not only logarithms of N, but also contributions proportional to γ_E and to $\zeta(n)$. Let us examine in more detail the first two classes of exponentiated constants.

As far as the ratio of form factors is concerned, one may use the result of ref. [7] showing that the absolute value of the ratio is finite to all orders and exponentiates. To illustrate these results, note that the (timelike) Sudakov form factor $\Gamma(Q^2, \epsilon)$ for the electromagnetic coupling of a massless quark of charge e_q is defined via

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_\mu(0) | p_1, p_2 \rangle = -\mathrm{i} e \, e_q \, \overline{v}(p_2) \gamma_\mu u(p_1) \; \Gamma\left(Q^2, \epsilon\right) , \tag{2.18}$$

with $Q^2 = (p_1 + p_2)^2$. Based on refs. [18, 19, 20], it was shown in ref. [7] that the dimensionally regularized Sudakov form factor may be written as an exponential of integrals over functions only of α_s,

$$\Gamma(Q^2, \epsilon) = \exp\left\{ \frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left[K\left(\alpha_s, \epsilon\right) + G\left(\overline{\alpha}\left(\xi^2\right), \epsilon\right) + \frac{1}{2} \int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K\left(\overline{\alpha}\left(\lambda^2\right)\right) \right] \right\} . \tag{2.19}$$

The function K in eq. (2.19) is defined to consist of counterterms only, while G is finite for $\epsilon \to 0$. Double logarithms of the hard scale Q arise from the double integral over the anomalous dimension $\gamma_K(\alpha_s)$. Note that within the framework of dimensional regularization all integrals in eq. (2.19) can be explicitly performed using the d-dimensional running coupling [21], so that the form factor can be expressed in terms of RG-invariant analytic functions of α_s and ϵ to any order.

[1] Concerning the three-loop coefficient $A^{(3)}$, the n_f-dependent part is known exactly [15, 16], while good numerical estimates exist for the n_f-independent term [17].

Using eq. (2.19) one can derive a particularly simple expression for the absolute value of the ratio appearing in eq. (2.15). One finds

$$\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right| = \exp\left\{\left|\frac{\mathrm{i}}{2}\int_0^\pi \left[G\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\theta}Q^2\right),\epsilon\right) - \frac{\mathrm{i}}{2}\int_0^\theta d\phi\, \gamma_K\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\phi}Q^2\right)\right)\right]\right|\right\}. \tag{2.20}$$

Performing the scale integrals, at the two-loop level this yields

$$\begin{aligned}\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 &= 1 + \frac{\alpha_s}{\pi}\,\frac{3\zeta(2)\gamma_K^{(1)}}{2} + \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{9}{8}\zeta^2(2)\left(\gamma_K^{(1)}\right)^2 + \frac{3}{4}\zeta(2)b_0G^{(1)}(0) + \frac{3}{2}\zeta(2)\gamma_K^{(2)}\right],\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}\gamma_K^{(1)} &= 2\,C_F\,,\\ G^{(1)}(\epsilon) &= C_F\left(\frac{3}{2} - \frac{\epsilon}{2}\left[\zeta(2) - 8\right] + \epsilon^2\left[8 - \frac{3}{4}\zeta(2) - \frac{7}{3}\zeta(3)\right] + \mathcal{O}(\epsilon^3)\right),\\ \gamma_K^{(2)} &= C_AC_F\left(\frac{67}{18} - \zeta(2)\right) - n_fC_F\left(\frac{5}{9}\right) \;=\; 2A^{(2)}\,,\end{aligned} \tag{2.22}$$

while $b_0 = (11C_A - 2n_f)/3$. The anomalous dimension γ_K is the "cusp" anomalous dimension of a Wilson line in the $\overline{\mathrm{MS}}$ renormalization scheme [22, 23, 24].

Eq. (2.21) illustrates the potential relevance of exponentiation of N-independent terms: first of all, the two-loop contribution is numerically dominated by one-loop effects, both through exponentiation and the running of the coupling (the first two terms of the two-loop coefficient numerically make up roughly three quarters of the total); furthermore, for this particular ratio, genuine two-loop effects are given only in terms of $\gamma_K^{(2)}$, thus they are UV-dominated and much simpler to calculate than the full form factor.

Finally, the function $F_{DIS}(\alpha_s)$ collects constant terms arising from phase space integrations in the various functions involved in the factorization, as well as from the exponentiation of the matching function H_{DIS}. One finds at one loop

$$F_{DIS}^{(1)} = C_F\left(\frac{1}{2} + \zeta(2)\right), \tag{2.23}$$

while at two loops some terms can be predicted by taking into account the running of the coupling, which yields

$$F_{DIS}^{(2)} = -\frac{3}{16}C_Fb_0\left(4 + \zeta(2) - 2\zeta(3)\right) + \delta F_{DIS}^{(2)}\,. \tag{2.24}$$

These two-loop contributions should not be taken too literally since, as indicated in eq. (2.24), there is at this level an uncalculated contribution $\delta F_{DIS}^{(2)}$ arising from a pure two-loop calculation, which could easily overwhelm the effects which have been included. Further discussion of the impact of these two-loop effects is given in section 4.

3. Exponentiation in the $\overline{\mathrm{MS}}$ scheme

As remarked in the introduction, one should not expect that the constants associated with the Sudakov form factor in the $\overline{\mathrm{MS}}$ -scheme Drell-Yan cross section can be organized as in the previous section, in terms of a simple ratio of the timelike to spacelike versions of the same form factor. The reason, of course, is that the $\overline{\mathrm{MS}}$ quark distribution function is not directly related to a physical process involving quark electromagnetic scattering at lowest order. We will show that it is nevertheless possible to organize these constants in a closely related manner.

Mass factorization of the Drell-Yan cross section in the $\overline{\mathrm{MS}}$ scheme is straightforward in moment space: one simply divides $\omega(N,\epsilon)$ by $\phi^2_{\overline{\mathrm{MS}}}(N,\epsilon)$, the square of the $\overline{\mathrm{MS}}$ quark distribution, defined by

$$\phi_{\overline{\mathrm{MS}}}(N,\epsilon) = \exp\left[\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left\{\int_0^1 dz\, \frac{z^{N-1}-1}{1-z} A\left(\overline{\alpha}\left(\xi^2\right)\right) + B_\delta\left(\overline{\alpha}\left(\xi^2\right)\right)\right\}\right] + \mathcal{O}\left(\frac{1}{N}\right), \tag{3.1}$$

with A the same function appearing in eq. (2.15), while B_δ is the virtual part of the non-singlet quark-quark splitting function ($B_\delta(\alpha) = B(\alpha)$ at one loop). As appropriate for an $\overline{\mathrm{MS}}$ parton density, one can easily verify that $\phi_{\overline{\mathrm{MS}}}(N,\epsilon)$ in eq. (3.1) is a series of pure counterterms. Q is the factorization scale, which for simplicity throughout this paper we set equal to the Drell-Yan invariant mass.[2] We will now factor this density into virtual and real parts

$$\phi_{\overline{\mathrm{MS}}}(N,\epsilon) = \phi_V(\epsilon)\,\phi_R(N,\epsilon)\,, \tag{3.2}$$

in such a way that in the (finite) ratio

$$\widehat{\omega}_{\overline{\mathrm{MS}}}(N) \equiv \frac{\omega(N,\epsilon)}{\phi_{\overline{\mathrm{MS}}}(N,\epsilon)^2} = \left(\frac{|\Gamma(Q^2,\epsilon)|^2}{\phi_V(\epsilon)^2}\right)\left(\frac{\psi_R(N,\epsilon)^2\, U_R(N,\epsilon)}{\phi_R(N,\epsilon)^2}\right) + \mathcal{O}\left(\frac{1}{N}\right) \tag{3.3}$$

the ratios of virtual functions and of real functions, displayed in the large brackets, are separately finite. To be precise, the factorization in eq. (3.2) is uniquely defined by the following criteria: first, the ratio of virtual functions must be finite; second, as we are factorizing a series of pure counterterms, we would like also $\phi_V(\epsilon)$ to consist only of poles. The real part $\phi_R(N,\epsilon)$ is then defined by eq. (3.2). Note that $\phi_V(\epsilon)$ defined in this way is process-dependent, in contrast to $\phi_{\overline{\mathrm{MS}}}(N,\epsilon)$; note also that, while $\phi_{\overline{\mathrm{MS}}}$ has only simple poles of collinear origin, the real and virtual contributions will have cancelling double poles. We will now analyze separately the real and virtual contributions to the cross section.

3.1 Cancellation of virtual poles

The timelike Sudakov form factor has imaginary parts, which are the source of the largest contributions to the ratio in eq. (2.20), while the $\overline{\mathrm{MS}}$ distribution is real. We can thus simplify our analysis by writing

$$\frac{|\Gamma(Q^2,\epsilon)|^2}{\phi_V(\epsilon)^2} = \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2\left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(\epsilon)}\right)^2. \tag{3.4}$$

[2] It is straightforward to repeat the analysis below keeping these scales different.

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The benefit of this lies in the fact that the first factor on the right hand side is finite, and already explicitly resummed in eq. (2.20). The second factor, on the other hand, is purely real. Inspired by the explicit expression for the form factor, eq. (2.19), we try the ansatz

$$\phi_V(\epsilon) = \exp\left\{\frac{1}{2}\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left[K\left(\alpha_s,\epsilon\right) + \widetilde{G}\left(\overline{\alpha}\left(\xi^2\right)\right) + \frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\lambda^2\right)\right)\right]\right\}, \qquad (3.5)$$

which has the same structure as the Sudakov form factor in eq. (2.19), with the difference that $\widetilde{G}$ has no order ϵ terms.

We will now show, to all orders in the strong coupling, that the perturbative coefficients of the function $\widetilde{G}$ can be chosen so as to render eq. (3.4) finite, and we will provide for them an explicit construction. Since the first ratio on the right hand side of eq. (3.4) is finite, it is sufficient to prove the cancellation of poles for the second ratio, $\Gamma(-Q^2,\epsilon)/\phi_V(\epsilon)$. First of all, one observes that all divergences arising from the terms K and γ_K manifestly cancel between $\Gamma(-Q^2,\epsilon)$ and $\phi_V(\epsilon)$, since they come from limits of integration independent of the particular energy scale considered. Hence divergences could only be generated by the G terms. To show that those terms can be made finite as well, we begin by writing

$$G\left(\alpha_s,\epsilon\right) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} G_m^{(n)}\epsilon^m\left(\frac{\alpha_s}{\pi}\right)^n, \qquad (3.6)$$

and by noting that the integration over the energy scale can be rewritten as one over the running coupling making use of

$$\frac{d\mu^2}{\mu^2} = 2\,\frac{d\,\overline{\alpha}}{\beta\left(\overline{\alpha},\epsilon\right)} = -\frac{1}{\epsilon}\frac{d\overline{\alpha}}{\overline{\alpha}}\frac{1}{1+\frac{1}{4\epsilon}\sum\limits_{m=1}^{\infty} b_{m-1}\left(\frac{\overline{\alpha}}{\pi}\right)^m}. \qquad (3.7)$$

To proceed, let us define the truncated perturbative expansions for the various functions involved by

$$\begin{aligned}
\widehat{\beta}_{(M)}(\alpha_s,\epsilon) &\equiv 1+\frac{1}{4\epsilon}\sum_{n=1}^{M} b_{n-1}\left(\frac{\alpha_s}{\pi}\right)^n, \\
G_{(M)}\left(\alpha_s,\epsilon\right) &\equiv \sum_{n=1}^{M}\sum_{m=0}^{\infty} G_m^{(n)}\epsilon^m\left(\frac{\alpha_s}{\pi}\right)^n, \\
\widetilde{G}_{(M)}\left(\alpha_s\right) &\equiv \sum_{n=1}^{M}\widetilde{G}^{(n)}\left(\frac{\alpha_s}{\pi}\right)^n.
\end{aligned} \qquad (3.8)$$

Finally, define

$$R_{(M)}^{(p)}(\alpha_s,\epsilon) \equiv \frac{G_{(M)}\left(\alpha_s,\epsilon\right) - \widetilde{G}_{(M)}\left(\alpha_s\right)}{\widehat{\beta}_{(M-1)}(\alpha_s,\epsilon)}, \qquad (3.9)$$

where the index (p) denotes the fact that the perturbative expansion for the ratio R is truncated at order α_s^p.

Writing down the integrands of both the numerator and the denominator of the second ratio in eq. (3.4) and keeping in mind the overall factor of $1/\epsilon$ from eq. (3.7), one can easily formulate our goal in terms of R. We need to show that $\widetilde{G}$ can be chosen so that

$$R^{(M)}_{(M)}(\alpha_s, \epsilon) = \mathcal{O}(\epsilon)\,, \qquad \forall M\,. \tag{3.10}$$

The proof proceeds by induction. First of all, one sees immediately that it is possible to get $R^{(1)}_{(1)}(\alpha_s, \epsilon) = \mathcal{O}(\epsilon)$, simply by choosing $\widetilde{G}^{(1)} = G^{(1)}_0$. Next we assume that eq. (3.10) holds for some fixed order M in perturbation theory for chosen values of $\widetilde{G}^{(n)}$, up to $n = M$, and show that $\widetilde{G}^{(M+1)}$ may be chosen so that eq. (3.10) holds also to order $M + 1$.

To isolate the terms of order α_s^{M+1} in $R^{(M+1)}_{(M+1)}$, we use the identity

$$\frac{1}{\widehat{\beta}_{(M)}(\alpha_s, \epsilon)} = \frac{1}{\widehat{\beta}_{(M-1)}(\alpha_s, \epsilon)} - \frac{\widehat{\beta}_{(M)}(\alpha_s, \epsilon) - \widehat{\beta}_{(M-1)}(\alpha_s, \epsilon)}{\widehat{\beta}_{(M)}(\alpha_s, \epsilon)\widehat{\beta}_{(M-1)}(\alpha_s, \epsilon)}\,. \tag{3.11}$$

Substituting eq. (3.11) into the definition of $R^{(M+1)}_{(M+1)}$, and neglecting all terms $\mathcal{O}(\alpha_s^{M+2})$ and higher, as well as terms which are explicitly $\mathcal{O}(\epsilon)$, one finds

$$R^{(M+1)}_{(M+1)}(\alpha_s, \epsilon) = R^{(M+1)}_{(M)}(\alpha_s, \epsilon) + \left(\frac{\alpha_s}{\pi}\right)^{M+1} \left(G^{(M+1)}_0 - \frac{b_{M-1}}{4} G^{(1)}_1 - \widetilde{G}^{(M+1)}\right). \tag{3.12}$$

The theorem is now proven if one can show that $R^{(M+1)}_{(M)}$ has no poles in ϵ, since a constant at $\mathcal{O}(\alpha_s^{M+1})$ can be removed by suitably defining $\widetilde{G}^{(M+1)}$. To see that this is the case, note that $R^{(M+1)}_{(M)}(\alpha_s, \epsilon)$ must, by the induction hypothesis, be of the form

$$R^{(M+1)}_{(M)}(\alpha_s, \epsilon) = \sum_{m=1}^{M} \sum_{l=1}^{\infty} c^{(m)}_l \epsilon^l \left(\frac{\alpha_s}{\pi}\right)^m + \left(\frac{\alpha_s}{\pi}\right)^{M+1} \sum_{p=p_0}^{\infty} d_p \epsilon^p\,. \tag{3.13}$$

Multiplying this times $\widehat{\beta}_{(M-1)}(\alpha_s, \epsilon)$, we must get back the numerator of $R^{(M+1)}_{(M)}$, up to $\mathcal{O}(\alpha_s^{M+2})$ corrections, namely

$$\begin{aligned} &\left[1 + \frac{1}{4\epsilon} \sum_{n=1}^{M-1} b_{n-1} \left(\frac{\alpha_s}{\pi}\right)^n\right] \cdot \left[\sum_{m=1}^{M} \sum_{l=1}^{\infty} c^{(m)}_l \epsilon^l \left(\frac{\alpha_s}{\pi}\right)^m + \left(\frac{\alpha_s}{\pi}\right)^{M+1} \sum_{p=p_0}^{\infty} d_p \epsilon^p\right] = \\ &= \sum_{n=1}^{M} \sum_{m=0}^{\infty} G^{(n)}_m \epsilon^m \left(\frac{\alpha_s}{\pi}\right)^n - \sum_{n=1}^{M} \widetilde{G}^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n + \mathcal{O}(\alpha_s^{M+2})\,. \end{aligned} \tag{3.14}$$

Now, the right hand side has no poles in ϵ, and no term of order α_s^{M+1}, thus on the left hand side all poles and all terms of order α_s^{M+1} must cancel. If $p_0 < 0$, on the other hand, one will generate a term of the form $d_{-1}(\alpha_s/\pi)^{(M+1)}(1/\epsilon)$ on the left hand side, which cannot be cancelled: in fact, note that the first sum in $R^{(M+1)}_{(M)}$ starts at $\mathcal{O}(\epsilon)$ by the induction hypothesis, and $\widehat{\beta}_{M-1}$ has only a simple pole in ϵ. We conclude that $p_0 \geq 0$, as desired.

To find explicit expressions one has to implement the observation that the coefficient of $\{(\alpha_s/\pi)^{M+1}, \epsilon^0\}$ in $R^{(M+1)}_{(M+1)}$ must vanish. One verifies that $\widetilde{G}^{(M+1)}$ is given by the coefficient of $\{(\alpha_s/\pi)^{M+1}, \epsilon^0\}$ in the ratio

$$\mathcal{R}_{(M+1)}(\alpha_s,\epsilon) \equiv \frac{G_{(M+1)}(\alpha_s,\epsilon)}{\widehat{\beta}_{(M)}(\alpha_s,\epsilon)}. \tag{3.15}$$

Explicitly,

$$\widetilde{G}^{(M+1)} = G_0^{(M+1)} - \frac{b_0}{4}G_1^{(M)} - \frac{b_1}{4}G_1^{(M-1)} + \frac{b_0^2}{16}G_2^{(M-1)} - \frac{b_2}{4}G_1^{(M-2)} + \frac{b_0 b_1}{8}G_2^{(M-2)} + \cdots. \tag{3.16}$$

3.2 Real emission contributions

The complete expression for the $\overline{\rm MS}$ -scheme DY cross section in the present framework is given by

$$\widehat{\omega}_{\overline{\rm MS}}(N) = \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(\epsilon)}\right)^2 \left[U_R(N,\epsilon)\left(\frac{\psi_R(N,\epsilon)}{\phi_R(N,\epsilon)}\right)^2\right], \tag{3.17}$$

where the factor in square brackets arises from real gluon emission. Each function appearing in the real emission contribution exponentiates: ψ_R and U_R according to eqs. (2.5) and (2.6), respectively, while ϕ_R is defined as the ratio of eq. (3.1) to eq. (3.5). Renormalization group arguments can be applied to each function, in $d = 4 - 2\epsilon$ dimensions, as described in the appendix. It is interesting to notice that running the coupling in d dimensions generates poles at two loops in the ratio $U_R\psi_R/\phi_R$, although the input at one loop is finite. The ratio must however be finite to all orders, as a consequence of the factorization theorem, together with the finiteness of the virtual contributions demonstrated above. This poses constraints on the two-loop coefficients of the functions involved, as described in more detail in section 4, tying together real and virtual contributions to the cross section.

Collecting and organizing the exponential expressions of the real emission functions, one can cast eq. (3.17) in the standard form

$$\begin{aligned}\widehat{\omega}_{\overline{\rm MS}}(N) = & \left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 \left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(\epsilon)}\right)^2 \exp\left[F_{\overline{\rm MS}}(\alpha_s)\right] \times \\ & \times \exp\left[\int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left\{2\int_{Q^2}^{(1-z)^2Q^2}\frac{d\mu^2}{\mu^2}A\left(\alpha_s(\mu^2)\right) + \right.\right. \\ & \left.\left. + D\left(\alpha_s\left((1-z)^2Q^2\right)\right)\right\}\right] + \mathcal{O}\left(\frac{1}{N}\right).\end{aligned} \tag{3.18}$$

A one loop calculation, with the inclusion of running coupling effects, yields

$$\begin{aligned}\log\left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(\epsilon)}\right) &= \frac{\alpha_s}{\pi}C_F\left(\frac{\zeta(2)}{4}-2\right) + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{C_F b_0}{2}\left(1-\frac{3}{32}\zeta(2)-\frac{7}{24}\zeta(3)\right)+\delta R^{(2)}\right], \\ F_{\overline{\rm MS}}(\alpha_s) &= \frac{\alpha_s}{\pi}C_F\left(-\frac{3}{2}\zeta(2)\right) + \left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{1}{4}b_0 C_F\left(1-\frac{3}{8}\zeta(2)-\frac{7}{4}\zeta(3)\right)+\delta F^{(2)}_{\overline{\rm MS}}\right],\end{aligned} \tag{3.19}$$

where $\delta R^{(2)}$ and $\delta F^{(2)}_{\overline{\rm MS}}$ are genuine two-loop contributions, formally unrelated to running coupling effects. Note that the function D in eq. (3.18) is the same as in eq. (2.15): a non-trivial statement, due to the fact that such a function, summarizing wide angle soft radiation, can be taken to vanish in the threshold-resummed deep-inelastic structure function [15, 25, 26, 27]. The coefficients $\delta F^{(2)}_{\overline{MS}}$ in eq. (3.19) and $\delta F^{(2)}_{DIS}$ in eq. (2.24) can be determined, if desired, by comparison with the exact two-loop results of ref. [8], in a manner similar to what was done in ref. [28].

We can now turn to a discussion of the relevance of the exponentiation of constants in the two schemes, as tested at the two-loop level.

4. Usage and limits of exponentiation

It is clear that the results we have derived on the exponentiation of N-independent terms do not have the predictive strength of the standard resummation of threshold logarithms. In that case, in fact, the pattern of exponentiation is highly nontrivial, and the perturbative coefficients of entire classes of logarithms can be exactly predicted to all perturbative orders performing just a low order calculation. In the present case, even though constant terms exponentiate, the determination of the exact value of the N-independent contribution at, say, g loops, always requires a g-loop calculation, albeit in some cases a simplified one.

It remains true, however, that exponentiation and running coupling effects generate contributions to all orders which originate from low-order calculations. These contributions are an easily computable subset of higher order corrections, and it is reasonable to use them to estimate the full impact of higher orders. Specifically, given a g-loop calculation of one of the cross sections we have discussed, one can extract the value of the various functions appearing in the exponent to that order, and then use exponentiation and RG running to estimate the $(g+1)$-loop result. Given the existing results at two loops [29, 30], one could construct an estimate of the three-loop partonic cross section. Before embarking in such a calculation, it is however advisable to test the case $g = 1$, i.e. to use the two-loop results to verify the reliability of the method, by comparing the exact results with the estimate obtained by exponentiating the one-loop calculation and letting the couplings run. To this end, we can expand the partonic cross section in scheme s, $\widehat{\omega}_s(N)$ as

$$\widehat{\omega}_s(N) = \sum_{p=0}^{\infty} \omega_s^{(p)}(N) \left(\frac{\alpha_s}{\pi}\right)^p . \tag{4.1}$$

Next, we identify the coefficients of different powers of $\log N$, by writing[3]

$$\omega_s^{(p)}(N) = \sum_{i=0}^{2p} \omega_{s,i}^{(p)} \left(\log N + \gamma_E\right)^i . \tag{4.2}$$

[3]Note that one could equivalently expand the partonic cross sections $\widehat{\omega}_s(N)$ in terms of linear combinations of Mellin transforms of the distributions $\delta(1-z)$ and $\mathcal{D}_i(z) \equiv \left(\log^i(1-z)/(1-z)\right)_+$. This would not significantly change the outlook of the discussion below.

i	0	1	2	3	4
DIS	0.26	1.17	0.13	0	0
$\overline{\mathrm{MS}}$	−0.69	1.79	0.33	0	0

Table 1: The deviations $\Delta\omega_{s,i}^{(2)}$, as defined in the text, for the DIS and $\overline{\mathrm{MS}}$ schemes.

So far we are dealing with the exact cross sections. Let now $\widetilde{\omega}_{s,i}^{(p)}$ be the estimate for $\omega_{s,i}^{(p)}$ obtained by evaluating the exponent exactly at $p-1$ loops, adding running coupling effects, and expanding the result to order p. One can define the deviation

$$\Delta\omega_{s,i}^{(p)} \equiv \frac{\omega_{s,i}^{(p)} - \widetilde{\omega}_{s,i}^{(p)}}{\omega_{s,i}^{(p)}} \, . \tag{4.3}$$

In computing our estimates for the DIS scheme, we will employ eq. (2.15), with the one-loop results taken from eq. (2.17), eq. (2.21) and eq. (2.23). Furthermore, since we are taking into account running coupling effects, we will include the terms proportional to b_0 in eqs. (2.24) and (A.8). As far as the $\overline{\mathrm{MS}}$ scheme is concerned, we will make use of eq. (3.18), together with eq. (3.19), with all purely two-loop contributions defined to vanish. Exact results are taken from refs. [29, 30], focusing on 'soft and virtual' contributions (all other contributions are suppressed by powers of N at large-N). To get numerical results we focus on SU(3) and set $n_f = 5$. The results for the deviations $\Delta\omega_{s,i}^{(2)}$ in the two schemes are given in table 1. To gain a little further insight, we also separate the contributions proportional to the possible combinations of group invariants arising at two loops (i.e. C_F^2, $C_A C_F$ and $n_f C_F$), and display the results in table 2 for the powers of $\log N$ which do not lead to exact agreement (i.e. $i = 0, 1, 2$). Here we give the coefficients separately for $\omega_{s,i}^{(2)}$ and $\widetilde{\omega}_{s,i}^{(2)}$ for each scheme, since some of the exact coefficients vanish, so that the corresponding deviations as given in eq. (4.3) are ill-defined.

Several remarks are in order. From table 1, we observe that, as expected, leading ($i = 4$) and next-to-leading ($i = 3$) logs at two loops are exactly predicted by one-loop results and running coupling effects. Similarly, also as expected, NNL logarithms (in this context $i = 2$) have a small discrepancy which is entirely traceable to the two-loop cusp anomalous dimension $\gamma_K^{(2)}$. At the level of N-independent terms ($i = 0$), the agreement is quite reasonable, and in fact rather satisfactory in the DIS scheme, where exponentiation accounts for three quarters of the exact answer.

Single log results, on the other hand, are much less satisfactory, displaying a discrepancy larger than 100% in both schemes. The reasons for this discrepancy are slightly different in the two schemes, but they highlight the same generic problem. In the DIS scheme, as discussed in the appendix, running coupling effects in the ratio of parton distributions ψ/χ generate a single log term with the wrong sign with respect to the exact result. This term must then be compensated by contributions which we would describe as 'genuine two-loop', which as a result display a rough proportionality to b_0.

This phenomenon could be described as an 'excess of factorization', in the following sense: to achieve the accuracy and generality of eq. (2.15) it is necessary to introduce several functions, depending on different scales. Not all of these dependencies, however, are

	DIS		$\overline{\text{MS}}$	
	estimate	exact	estimate	exact
coefficients of C_F^2				
i = 0	38.06	39.03	3.33	1.79
i = 1	−13.09	−14.41	0	0
i = 2	9.85	9.85	5.16	5.16
coefficients of $C_A C_F$				
i = 0	5.63	18.12	13.12	6.82
i = 1	9.83	−0.25	1.51	−0.47
i = 2	−0.69	0.35	0	2.08
coefficients of $n_f C_F$				
i = 0	−1.02	−4.40	−2.38	−0.80
i = 1	−1.79	0.35	−0.27	−0.52
i = 2	0.12	−0.15	0	−0.56

Table 2: Comparison between estimates from exponentiation and exact results at two loops, presented separately for different colour structures, both in the DIS and $\overline{\text{MS}}$ schemes.

physical, and there may be (in fact there are) large cancellations in the scale dependence between different functions. This fact has been observed in the past: Catani and Trentadue pointed it out in ref. [31], and more recently a similar observation was made by Gardi and Roberts in ref. [27]. As a consequence, approximate coefficients that are dominated by running coupling effects, but not completely determined, may turn out to be quite inaccurate.

In the $\overline{\text{MS}}$ scheme, the same kind of cancellation is displayed in a different way: there, as described in more detail below, one may use the constraint imposed by the finiteness of the real emission contributions to reexpress the single log coefficient in terms of purely virtual functions, and in the process the weight of running coupling effects changes considerably. Again, this indicates that computed running coupling effects may easily be compensated by unevaluated two-loop contributions.

Finally, one may observe in table 2 that abelian (C_F^2) contributions exponentiate with impressive accuracy, particularly in the DIS scheme. The slight superiority of the DIS scheme in this regard appears to be a fairly generic feature in tables 1 and 2, perhaps to be ascribed to the more direct physical interpretation of the subtractions, as compared to the $\overline{\text{MS}}$ scheme.

To conclude, we would like to point out that the methods of exponentiation we have outlined may be used not only for numerical estimates, but also to obtain, or test, analytical results. Specifically, since all functions employed have precise diagrammatic definitions, the computation of certain coefficients at two loops may be simplifed using this approach, as compared to a full calculation of the cross section. Further, the factorization into separately finite real and virtual contributions leads to constraints connecting different coefficients, so that different two-loop results can be nontrivially connected. To give an example, consider eq. (3.17). There, the finiteness of the ratio $U_R \psi_R / \phi_R$ at two loops imposes constraints tying together real and virtual contributions to the cross section (recall that ϕ_R is defined as $\phi_{\overline{\rm MS}}/\phi_V$). Using the methods outlined in the appendix and imposing the cancellation of double poles in the ratio of real functions, one may verify that the two-loop coefficient of $\log^2 N$ in $\widehat{\omega}_{\overline{\rm MS}}(N)$ must equal the two-loop cusp anomalous dimension $\gamma_K^{(2)}$, as is well known. Further, imposing the cancellation of single poles in the same ratio, one finds that the value of the function D at two loops is completely determined by purely virtual diagrams. One finds

$$D^{(2)} = \frac{3}{4}\zeta(2) b_0 C_F + 4 B_\delta^{(2)} - 2\widetilde{G}^{(2)}\,, \tag{4.4}$$

where $B_\delta^{(2)}$, the two-loop virtual part of the non-singlet quark-quark splitting function [32, 33], is given by

$$B_\delta^{(2)} = \frac{3}{2} C_F^2 \left(\frac{1}{16} - \frac{1}{2}\zeta(2) + \zeta(3)\right) + \frac{C_A C_F}{4}\left(\frac{17}{24} + \frac{11}{3}\zeta(2) - 3\zeta(3)\right) - \frac{n_f C_F}{6}\left(\frac{1}{8} + \zeta(2)\right), \tag{4.5}$$

while the second order contribution to the function $\widetilde{G}$ can be determined via eqs. (3.16) and (2.19), by matching the resummed expression for $\Gamma(Q^2, \epsilon)$ to the explicit results for the dimensionally regularized one- and two-loop Sudakov form factors, as given for example in ref. [8, 34, 35]. We find

$$\begin{aligned}
\widetilde{G}^{(2)} &= G_0^{(2)} - \frac{b_0}{4} G_1^{(1)}\,, \\
G_0^{(2)} &= 3\, C_F^2 \left(\frac{1}{16} - \frac{1}{2}\zeta(2) + \zeta(3)\right) - \frac{C_A C_F}{4}\left(13 - \frac{11}{3}\zeta(2) - \frac{2545}{108}\zeta(3)\right) - \\
&\quad - \frac{n_f C_F}{6}\left(\frac{209}{36} + \zeta(2)\right), \\
G_1^{(1)} &= C_F \left(4 - \frac{1}{2}\zeta(2)\right).
\end{aligned} \tag{4.6}$$

We have thus rederived the coefficient $D^{(2)}$, obtained earlier in refs. [13, 14] through matching to the two-loop cross sections of ref. [29, 30], by using only information from purely virtual contributions.

5. Conclusions

We have shown how to organize *all* constants in the N-th moment of the Drell-Yan cross section in the DIS and $\overline{\rm MS}$ schemes into exponential forms. Our $\overline{\rm MS}$ -scheme result has the special feature that real and virtual contributions are separately finite. This organization rests crucially upon the refactorization properties of the unsubtracted Drell-Yan cross section and, for the DIS scheme, of the non-singlet deep-inelastic structure function, near threshold [3]. For the $\overline{\rm MS}$ scheme the organization involves the construction of an exponential series of pure counterterms that cancels all divergences in the spacelike Sudakov form factor. We have proven this cancellation to all orders. We emphasize that our arguments imply exponentiation to the same degree of accuracy for the $\overline{\rm MS}$ -scheme DIS cross section $\widehat{F}_2(N)$, although we have not given a detailed evaluation in that case.

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Although exponentiation of N-independent terms does not have the same degree of predictive power as the resummation of threshold logarithms, it can be used with some degree of confidence to gauge the impact of higher order corrections to fixed order cross sections: we found that N-independent contributions at two loops are reasonably well approximated by the exponentiation of one-loop results. The refactorization approach also leads to nontrivial connections between real and virtual contributions to the cross section, which can be used to test or in some cases simplify finite order calculations. On the negative side, one cannot in general trust running coupling effects to give by themselves a good approximation of two-loop results, unless the various scales at which the couplings are evaluated are tied to the physical scales of the full cross section.

It might be interesting to make full use of the available two-loop information for the Drell-Yan and DIS cross sections to provide an estimate of three-loop effects, along the lines of section 4. We prefer to regard the techniques presented here, which extend threshold resummations to a new class of terms, as a step towards the analysis of yet other classes of perturbative corrections which might be expected to exponentiate. A natural example is given by threshold logarithms suppressed by an extra power of the Mellin variable N, which have recently been analized in the case of the longitudinal DIS structure function $F_L(N)$ in refs. [36, 37], and were shown to be phenomenologically important in ref. [38]. Another possible extension of our work is of course the study of N-independent contributions to more complicated hard cross sections, involving more colored particles, along the lines of refs. [39] and [40]: this would be an ingredient towards a precise resummed determination of such cross sections, which would be of considerable phenomenological interest for present and future hadron colliders.

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A. Evolving unconventional parton distributions

Let us briefly discuss the application of renormalization group techniques to unconventional parton distributions such as the functions $\psi(N, \epsilon)$ and $\chi(N, \epsilon)$ described in the text, specifically focusing on the contributions involving real gluon emission. The techniques of ref. [3] lead to an expression of the form

$$\psi_R(N, \epsilon) = \exp \left\{ \int_0^1 dz \, \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \, \kappa_\psi \left(\frac{(1-y)Q}{\mu}, \alpha_s(\mu^2), \epsilon \right) \right\} + \mathcal{O}\left(\frac{1}{N}\right), \quad \text{(A.1)}$$

and similarly for χ. The functions ψ_R and χ_R are both renormalization group invariant (i.e. their respective anomalous dimensions vanish) to this accuracy, for slightly different reasons: ψ_R cannot have overall UV divergences because its phase space is restricted to fixed total energy emitted in the final state. This automatically restrict also transverse momentum, so the phase space integration is UV finite. χ_R, on the other hand, has a phase space restricted to fixed total light-cone momentum fraction, so that in principle it may have UV divergences arising from transverse momentum integrations. These divergences are in fact present, however it can be shown that, at least at one loop and in the chosen axial gauge, these divergences are suppressed by powers of N. Note that this is not in contradiction with the fact that the divergent terms for any quark distribution must be proportional to the Altarelli-Parisi kernel. It simply means that the corresponding divergences are of IR-collinear origin for the distributions at hand.

The consequence of this statement for the functions κ_ψ and κ_χ is that

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\left(\epsilon, \alpha_s\right)\frac{\partial}{\partial\alpha_s}\right)\kappa_\psi\left(\frac{(1-y)Q}{\mu}, \alpha_s(\mu^2), \epsilon\right) = 0\,, \tag{A.2}$$

where $\beta\left(\epsilon, \alpha_s\right)$ is the β function in $d = 4 - 2\epsilon$. An identical equation is obeyed by κ_χ. Such equations can be solved perturbatively to determine the dependence of the distributions on the momentum scale. Consider again for example κ_ψ, and let $\xi \equiv (1-y)Q/\mu$. Expanding

$$\kappa_\psi\left(\xi, \alpha_s, \epsilon\right) = \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \kappa_\psi^{(n)}\left(\xi, \epsilon\right)\,, \tag{A.3}$$

one easily finds that at the two loop level the coefficients must be of the form

$$\begin{aligned}
\kappa_\psi^{(1)}\left(\xi, \epsilon\right) &= \kappa_{\psi,0}^{(1)}\left(\epsilon\right)\xi^{-2\epsilon}\,, \\
\kappa_\psi^{(2)}\left(\xi, \epsilon\right) &= \kappa_{\psi,0}^{(2)}\left(\epsilon\right)\xi^{-4\epsilon} + \frac{b_0}{4\epsilon}\kappa_{\psi,0}^{(1)}\left(\epsilon\right)\xi^{-2\epsilon}\left(\xi^{-2\epsilon} - 1\right)\,.
\end{aligned} \tag{A.4}$$

explicit evaluation at one loop [3] yields

$$\begin{aligned}
\kappa_{\psi,0}^{(1)}\left(\epsilon\right) &= 2C_F\left(4\pi\right)^\epsilon\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\,, \\
\kappa_{\chi,0}^{(1)}\left(\epsilon\right) &= 2C_F\left(4\pi\right)^\epsilon\Gamma(2+\epsilon)\cos\left(\pi\epsilon\right)\,.
\end{aligned} \tag{A.5}$$

As observed in section 2, the finiteness of the ratio ψ_R/χ_R, which is a consequence of factorization, requires that $\kappa_{\psi,0}^{(1)}(\epsilon) - \kappa_{\chi,0}^{(1)}(\epsilon) = \mathcal{O}(\epsilon^2)$, since the double integration in eq. (A.1) generates a double pole. In fact, upon redefing μ according to the $\overline{\rm MS}$ prescription to absorb factors of $\log(4\pi)$ and γ_E,

$$\kappa_{\psi,0}^{(1)}\left(\epsilon\right) - \kappa_{\chi,0}^{(1)}\left(\epsilon\right) = 2C_F\epsilon^2\left[2 + \zeta(2) + \epsilon\left(4 + \zeta(2) - 2\zeta(3)\right) + \mathcal{O}(\epsilon^2)\right]. \tag{A.6}$$

One can go slightly further and observe that the finiteness of the ratio ψ_R/χ_R also constrains the form of the pure two-loop contribution to κ_ψ and κ_χ, given by the functions $\kappa_{\psi,0}^{(2)}(\epsilon)$ and $\kappa_{\chi,0}^{(2)}(\epsilon)$. Specifically, inserting eqs. (A.3) and (A.4) into eq. (A.1), and doing the same for χ_R, one finds that the ratio ψ_R/χ_R will develop a simple pole in ϵ at two loops, unless

$$\kappa_{\psi,0}^{(2)}\left(\epsilon\right) - \kappa_{\chi,0}^{(2)}\left(\epsilon\right) = \frac{3}{2}C_F b_0\epsilon\left(2 + \zeta(2)\right) + \epsilon^2\delta\kappa_2^{(2)} + \mathcal{O}\left(\epsilon^3\right)\,, \tag{A.7}$$

in analogy with eq. (A.6), with $\delta\kappa_2^{(2)}$ a constant arising at two loops to be used below. This constraint also fixes the coefficient of a contribution to the ratio proportional to $\log N$ at two loops, i.e. at NNL level. To be precise one finds

$$\left(\frac{\psi_R(N,\epsilon)}{\chi_R(N,\epsilon)}\right)^2 = \exp\left[\frac{\alpha_s}{\pi}C_F\,(2+\zeta(2)) + \right.$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{1}{8}\delta\kappa_2^{(2)} + \frac{1}{2}C_F b_0\,(2+\zeta(2))\,(\log N + \gamma_E) - \right.$$
$$\left.\left. - \frac{3}{16}C_F b_0\,(4+\zeta(2)-2\zeta(3))\right) + \mathcal{O}\left(\epsilon, \frac{1}{N}, \alpha_s^3\right)\right]. \quad \text{(A.8)}$$

Once again, the contributions arising at two loops should be taken with a grain of salt when constructing the full cross section. It is true in fact that in this way we have determined the leading logarithmic contribution to this particular ratio, however in the full cross section there are competing $\log N$ terms arising at two loops from other functions, and in fact in the present case the logarithmic term in eq. (A.8) goes in the wrong direction to 'predict' NNL logarithms at two loops, as discussed in the text. Similarly, there is no guarantee that the uncalculated constant $\delta\kappa_2^{(2)}$ will not overwhelm the running coupling effects explicitly displayed in eq. (A.8).

References

[1] J.C. Collins, D.E. Soper and G. Sterman, *Factorization of hard processes in QCD*, in *Perturbative Quantum Chromodynamics*, A.H. Mueller ed., World Scientific, Singapore 1989.

[2] W. Giele et al., *The QCD/SM working group: summary report*, hep-ph/0204316.

[3] G. Sterman, *Summation of large corrections to short distance hadronic cross-sections*, *Nucl. Phys.* **B 281** (1987) 310.

[4] S. Catani and L. Trentadue, *Resummation of the QCD perturbative series for hard processes*, *Nucl. Phys.* **B 327** (1989) 323.

[5] G. Parisi, *Summing large perturbative corrections in QCD*, *Phys. Lett.* **B 90** (1980) 295.

[6] J.C. Collins, *Sudakov form factors*, in *Perturbative Quantum Chromodynamics*, A.H. Mueller ed., World Scientific, Singapore 1989.

[7] L. Magnea and G. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, *Phys. Rev.* **D 42** (1990) 4222.

[8] T. Matsuura, S.C. van der Marck and W.L. van Neerven, *The calculation of the second order soft and virtual contributions to the Drell-Yan cross-section*, *Nucl. Phys.* **B 319** (1989) 570.

[9] L. Magnea, *All order summation and two loop results for the Drell-Yan cross-section*, *Nucl. Phys.* **B 349** (1991) 703.

[10] L. Magnea and G. Sterman, *Resummation of large terms in hard scattering*, in Fermilab 1990, proceedings of *Workshop on hadron structure functions and parton distributions*, World Scientific, Singapore 1990.

[11] J.G.M. Gatheral, *Exponentiation of eikonal cross-sections in nonabelian gauge theories*, *Phys. Lett.* **B 133** (1983) 90.

JHEP06(2003)057

[12] J. Frenkel and J.C. Taylor, *Nonabelian eikonal exponentiation*, *Nucl. Phys.* **B 246** (1984) 231.

[13] H. Contopanagos, E. Laenen and G. Sterman, *Sudakov factorization and resummation*, *Nucl. Phys.* **B 484** (1997) 303 [hep-ph/9604313].

[14] A. Vogt, *Next-to-next-to-leading logarithmic threshold resummation for deep-inelastic scattering and the Drell-Yan process*, *Phys. Lett.* **B 497** (2001) 228 [hep-ph/0010146].

[15] S. Moch, J.A.M. Vermaseren and A. Vogt, *Non-singlet structure functions at three loops: fermionic contributions*, *Nucl. Phys.* **B 646** (2002) 181 [hep-ph/0209100].

[16] C.F. Berger, *Higher orders in $A(\alpha_s)/(1-x)_+$ of non-singlet partonic splitting functions*, *Phys. Rev.* **D 66** (2002) 116002 [hep-ph/0209107].

[17] W.L. van Neerven and A. Vogt, *Improved approximations for the three-loop splitting functions in QCD*, *Phys. Lett.* **B 490** (2000) 111 [hep-ph/0007362].

[18] A. Sen, *Asymptotic behavior of the Sudakov form-factor in QCD*, *Phys. Rev.* **D 24** (1981) 3281.

[19] A.H. Mueller, *On the asymptotic behavior of the Sudakov form-factor*, *Phys. Rev.* **D 20** (1979) 2037.

[20] J.C. Collins, *Algorithm to compute corrections to the Sudakov form-factor*, *Phys. Rev.* **D 22** (1980) 1478.

[21] L. Magnea, *Analytic resummation for the quark form factor in QCD*, *Nucl. Phys.* **B 593** (2001) 269 [hep-ph/0006255].

[22] J. Kodaira and L. Trentadue, *Summing soft emission in QCD*, *Phys. Lett.* **B 112** (1982) 66.

[23] G.P. Korchemsky and G. Marchesini, *Structure function for large x and renormalization of Wilson loop*, *Nucl. Phys.* **B 406** (1993) 225 [hep-ph/9210281].

[24] G.P. Korchemsky and A.V. Radyushkin, *Infrared factorization, Wilson lines and the heavy quark limit*, *Phys. Lett.* **B 279** (1992) 359 [hep-ph/9203222].

[25] S. Forte and G. Ridolfi, *Renormalization group approach to soft gluon resummation*, *Nucl. Phys.* **B 650** (2003) 229 [hep-ph/0209154].

[26] E. Gardi, G.P. Korchemsky, D.A. Ross and S. Tafat, *Power corrections in deep inelastic structure functions at large Bjorken x*, *Nucl. Phys.* **B 636** (2002) 385 [hep-ph/0203161].

[27] E. Gardi and R.G. Roberts, *The interplay between Sudakov resummation, renormalons and higher twist in deep inelastic scattering*, *Nucl. Phys.* **B 653** (2003) 227 [hep-ph/0210429].

[28] N. Kidonakis, *A unified approach to NNLO soft and virtual corrections in electroweak, Higgs, QCD and SUSY processes*, hep-ph/0303186.

[29] R. Hamberg, W.L. van Neerven and T. Matsuura, *A complete calculation of the order α_s^2 correction to the Drell-Yan K factor*, *Nucl. Phys.* **B 359** (1991) 343.

[30] W.L. van Neerven and E.B. Zijlstra, *The $\mathcal{O}(\alpha_s^2)$ corrected Drell-Yan k factor in the DIS and $\overline{\rm MS}$ scheme*, *Nucl. Phys.* **B 382** (1992) 11.

[31] S. Catani and L. Trentadue, *Comment on QCD exponentiation at large x*, *Nucl. Phys.* **B 353** (1991) 183.

[32] R.T. Herrod and S. Wada, *Altarelli-Parisi equation in the next-to-leading order*, *Phys. Lett.* **B 96** (1980) 195.

JHEP06(2003)057

[33] G. Curci, W. Furmanski and R. Petronzio, *Evolution of parton densities beyond leading order: the nonsinglet case*, *Nucl. Phys.* **B 175** (1980) 27.

[34] G. Kramer and B. Lampe, *Two jet cross-section in* e^+e^- *annihilation*, *Z. Physik* **C 34** (1987) 497.

[35] R.J. Gonsalves, *Dimensionally regularized two loop on-shell quark form-factor*, *Phys. Rev.* **D 28** (1983) 1542.

[36] R. Akhoury, M.G. Sotiropoulos and G. Sterman, *An operator expansion for the elastic limit*, *Phys. Rev. Lett.* **81** (1998) 3819 [hep-ph/9807330].

[37] R. Akhoury and M.G. Sotiropoulos, *The large-x factorization of the longitudinal structure function*, hep-ph/0304131.

[38] M. Kramer, E. Laenen and M. Spira, *Soft gluon radiation in Higgs boson production at the LHC*, *Nucl. Phys.* **B 511** (1998) 523 [hep-ph/9611272].

[39] N. Kidonakis and G. Sterman, *Resummation for QCD hard scattering*, *Nucl. Phys.* **B 505** (1997) 321 [hep-ph/9705234].

[40] G. Sterman and M.E. Tejeda-Yeomans, *Multi-loop amplitudes and resummation*, *Phys. Lett.* **B 552** (2003) 48 [hep-ph/0210130].

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